

Towards the Heider balance – standards and surprises

Krzysztof Kułakowski

in cooperation with

Zdzisław Burda, Przemysław Gawroński,

Małgorzata Krawczyk, Krzysztof Malarz

Maciej Wołoszyn

AGH University of Science and Technology
Faculty of Physics and Applied Computer Science
Kraków, Euroland

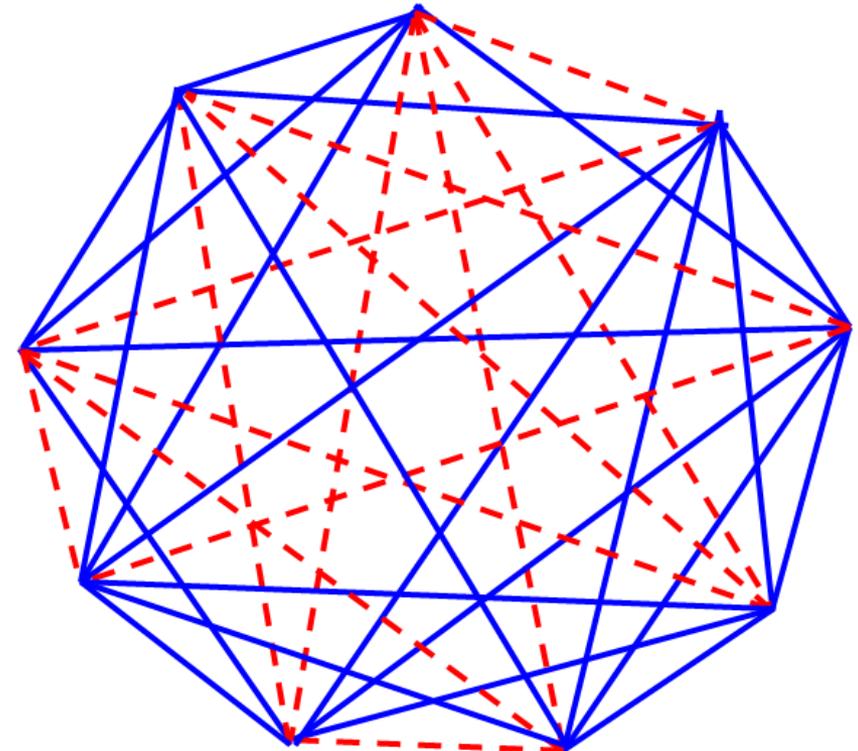


1-3 July 2021

Friendly and hostile relations distributed in a complete social network



Who is my enemy? Who is my friend?



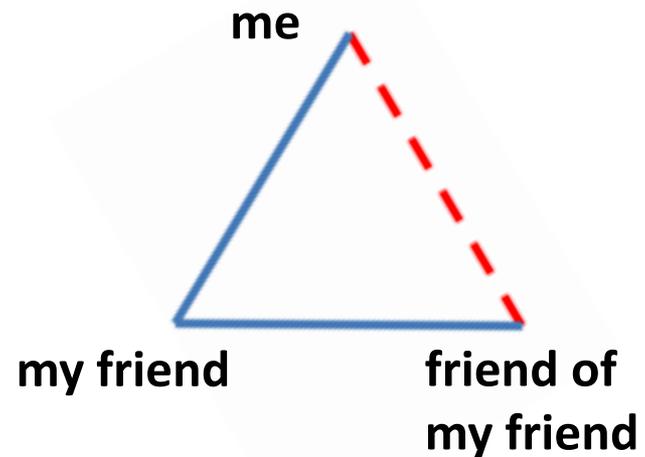
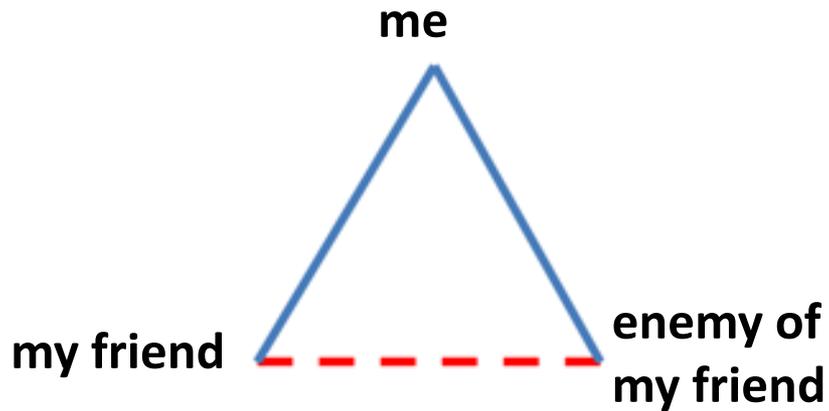
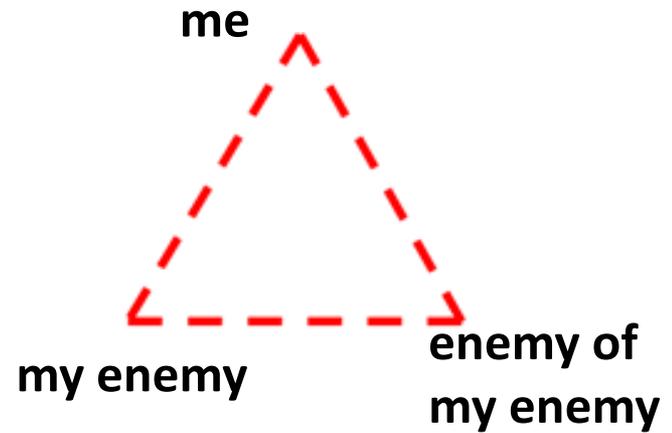
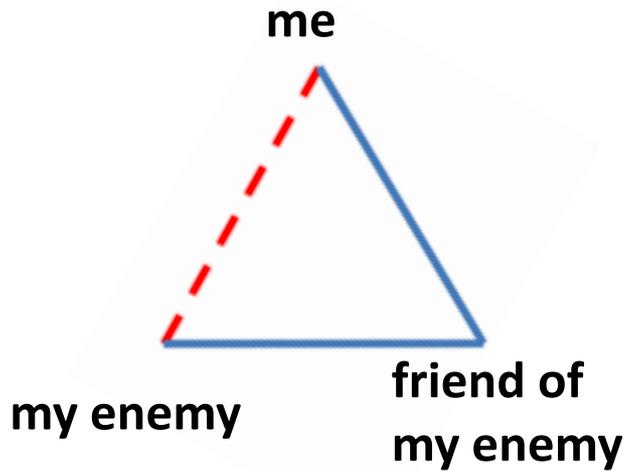
random configuration

IMBALANCED TRIADS

$$s(ij)s(jk)s(ki) < 0$$

----- hostile, negative

———— friendly, positive

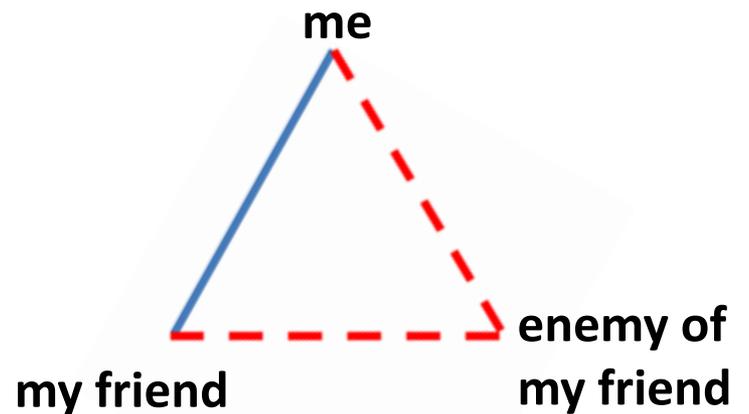
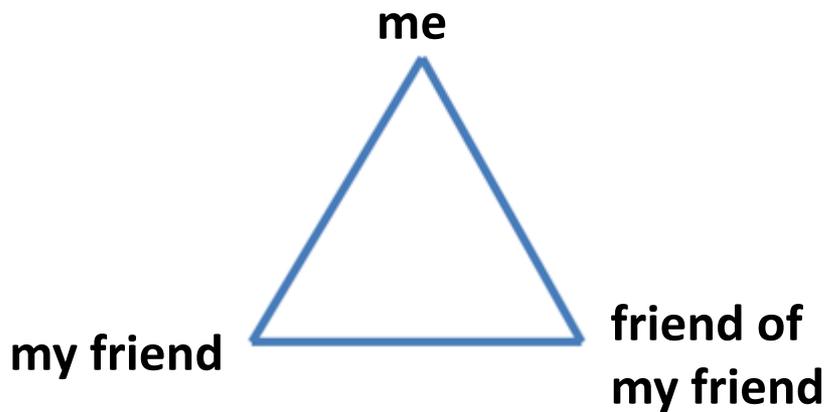
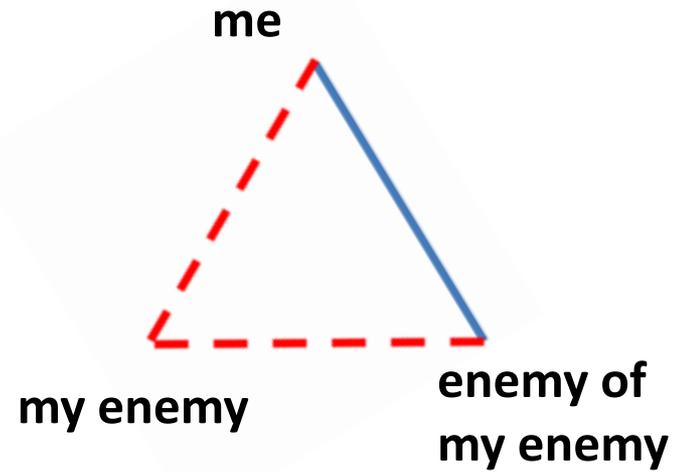
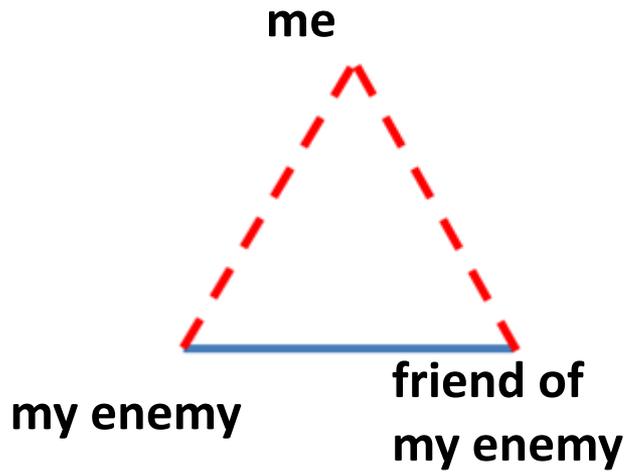


BALANCED TRIADS

$$s(ij)s(jk)s(ki) > 0$$

----- hostile, negative

———— friendly, positive



THE PSYCHOLOGICAL REVIEW

STRUCTURAL BALANCE: A GENERALIZATION OF HEIDER'S THEORY ¹

DORWIN CARTWRIGHT AND FRANK HARARY
Research Center for Group Dynamics, University of Michigan

Df: A graph is balanced
if all its cycles are positive.

Structure theorem: A graph is balanced
iff its points can be separated into two
mutually exclusive subsets such that
each positive line joins two points of the
same subset and each negative line joins
points from different subsets.



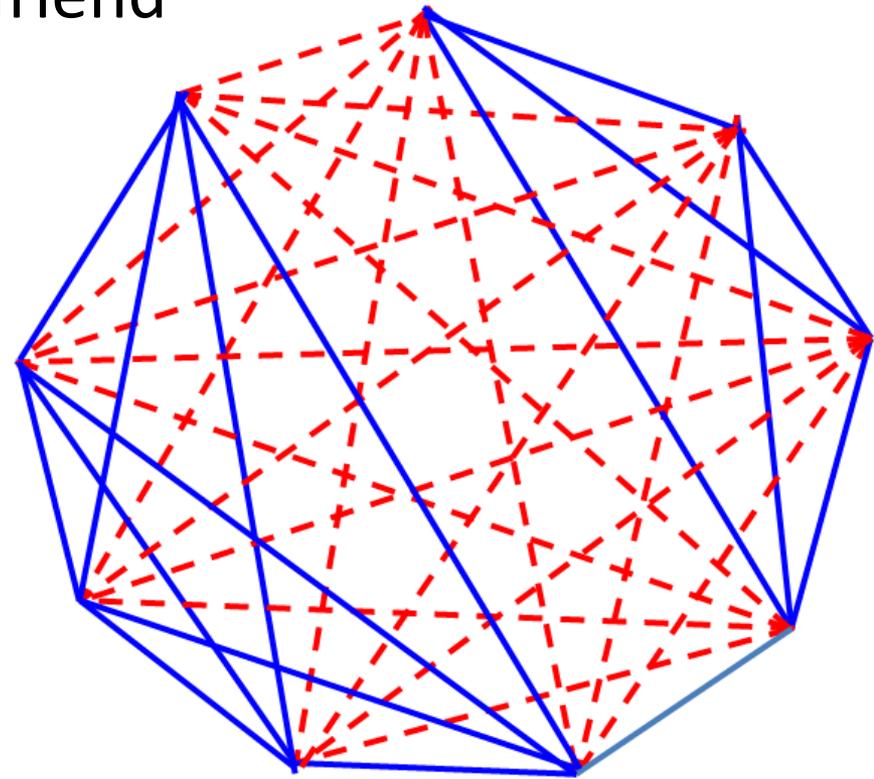
FH, mathematician



DPC, social psychologist

Dissonance about interpersonal relations is removed, if :

- A friend of my friend is my friend
- A friend of my enemy is my enemy
- An enemy of my friend is my enemy
- An enemy of my enemy is my friend



balanced configuration

outline

States of relations modelled by:

- heat bath simulation
- synchronous cellular automata, local and global
- sets of ordinary differential equations

New results:

- critical temperature vs topology
- synchronous dynamics: long cycles
- relations between opinions

end of
introduction

Tibor Antal



Paul Krapivsky



2 seminal papers:

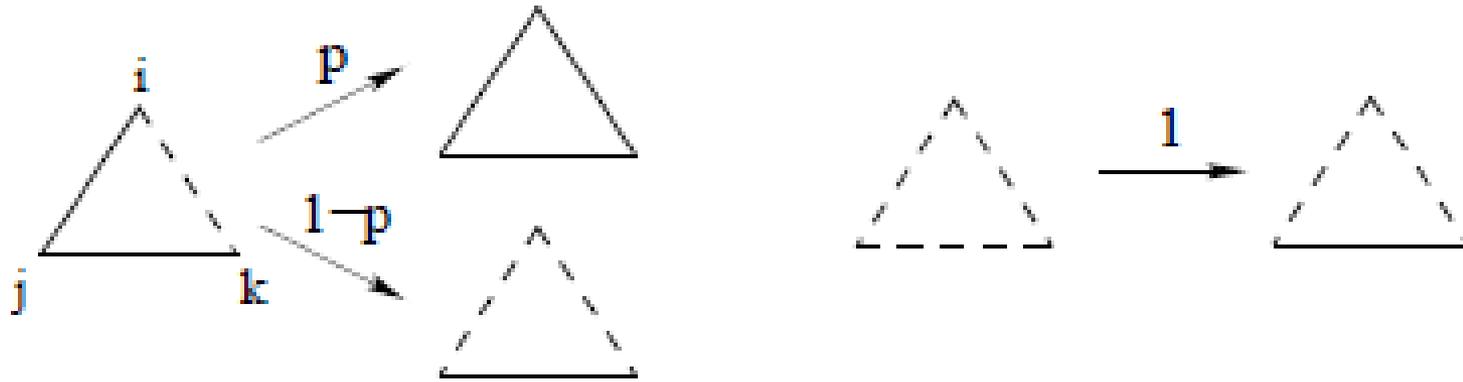
Dynamics of social balance on networks
Phys. Rev. E 72 (2005) 036121

*Social balance on networks:
The dynamics of friendship and enmity*
Physica D 224 (2006) 130

Sidney Redner
Kadanoff Prize 2021

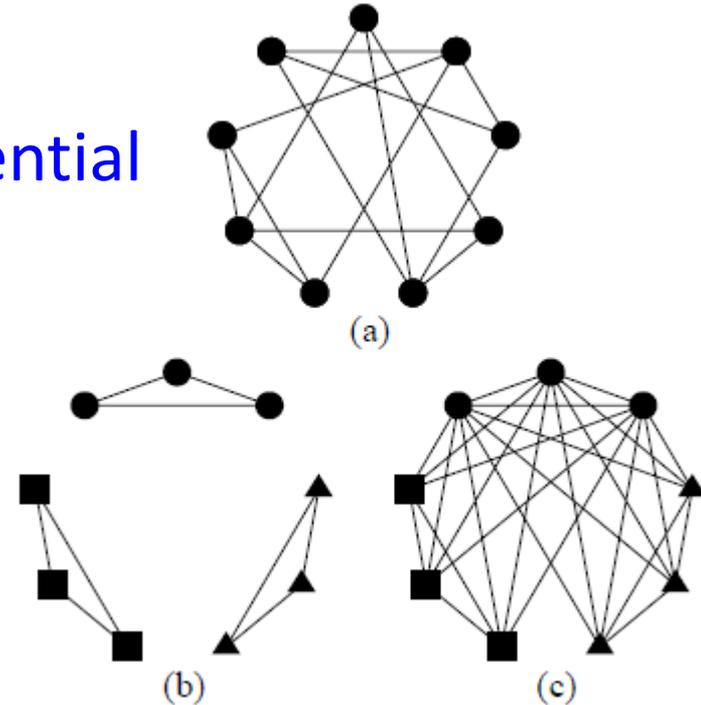


I. Local Triad Dynamics - sequential



II. Constrained Triad Dynamics- sequential

- select a link randomly,
- change its sign, if the total number of imbalanced triads decreases.



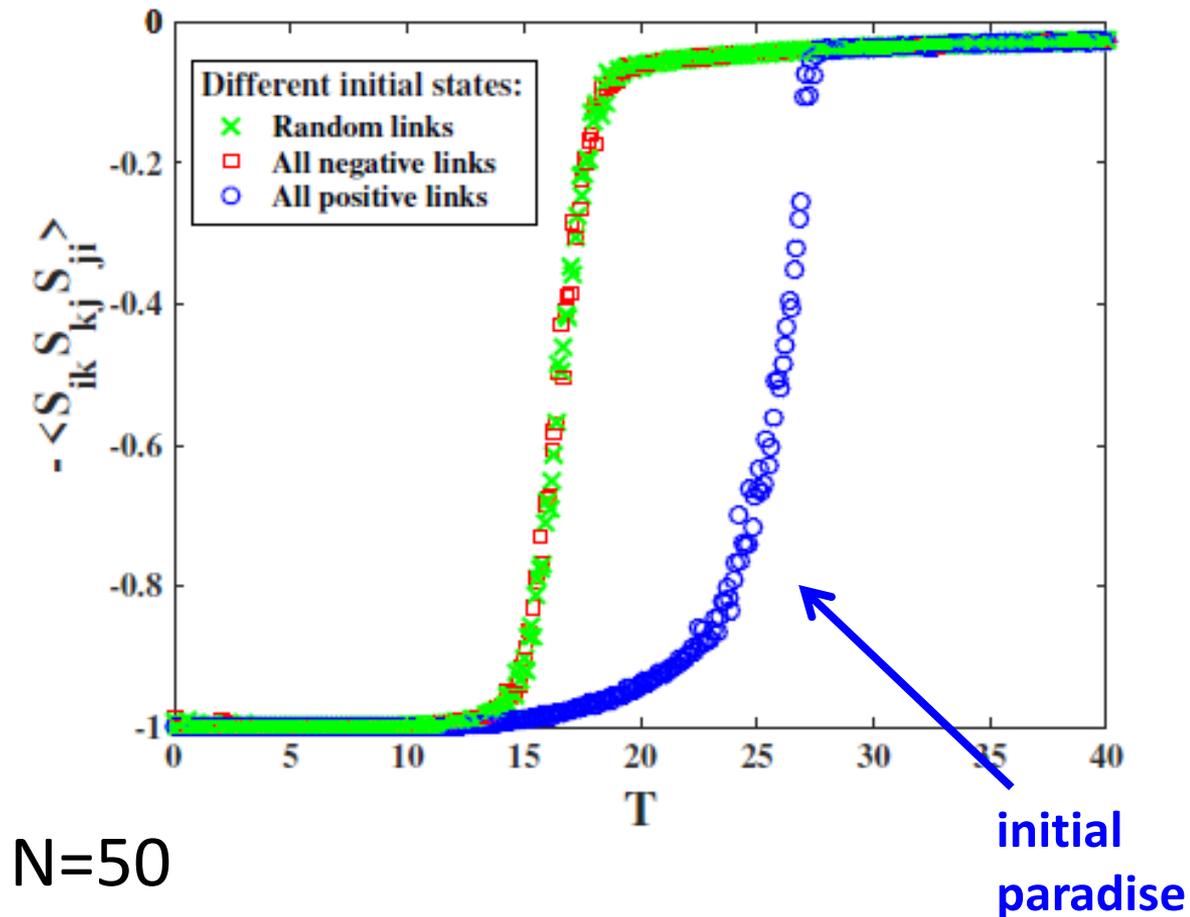
Examples of jammed states for $N=9$ - only positive links are shown.

The process to reduce the cognitive dissonance could minimize a work function – a kind of energy

$$U = - \sum_{i>j>k} s_{ij} s_{jk} s_{ki}$$

where $s_{ki} = \pm 1$ is the relation between k and i , friendly or hostile

a complete graph $T > 0$



F. Rabbani, A. Shirazi, G. Jafari, Phys. Rev. E 99 (2019), 062302

a complete graph

$\rho(t)$ = fraction of positive links vs time

effective field acting at $s(ij)$

$$-\frac{\partial U}{\partial s_{ij}} = \sum_k^N s_{ik} s_{kj}$$

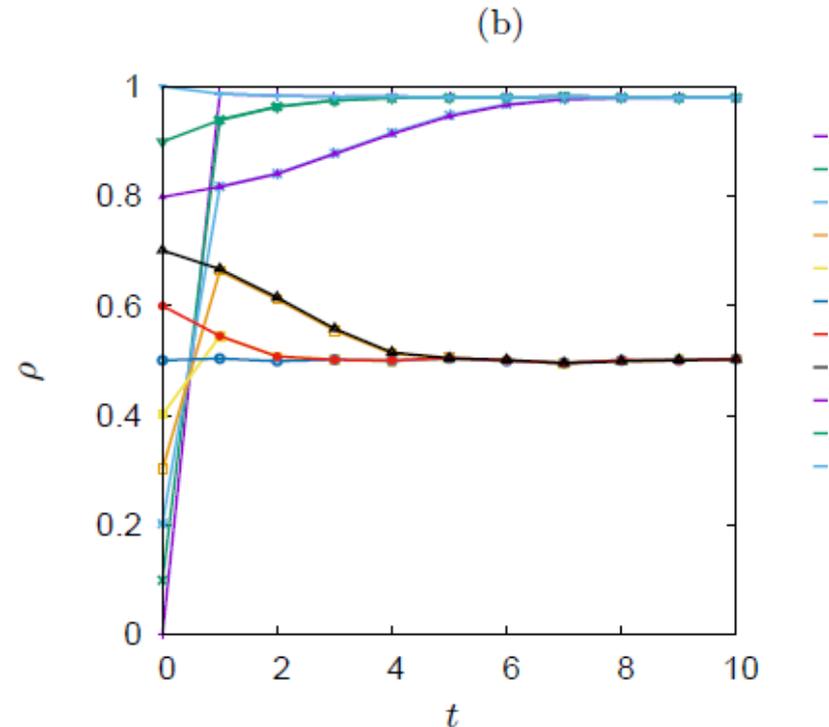
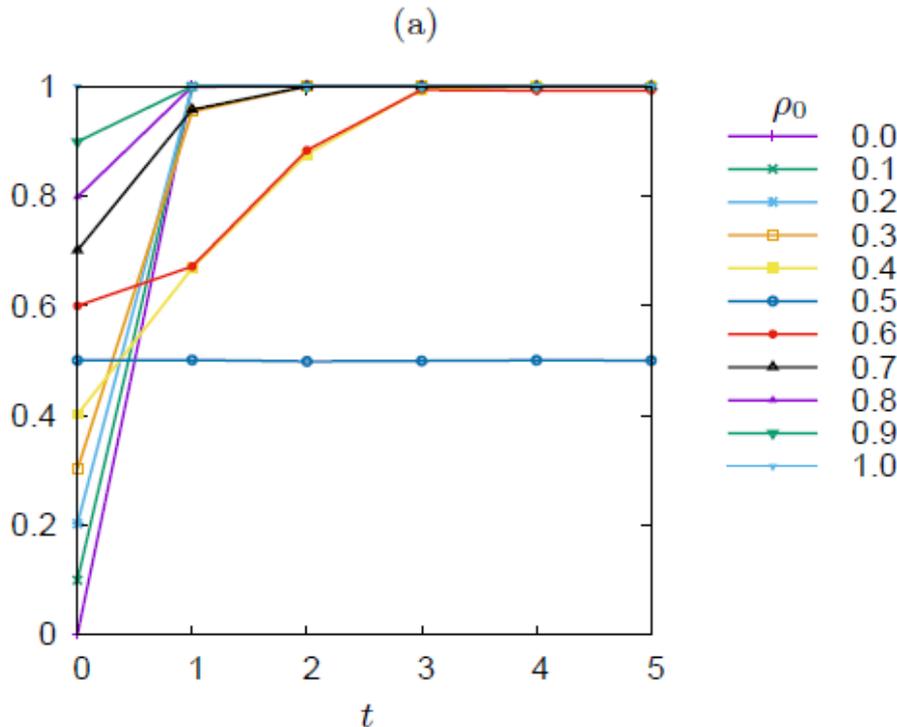
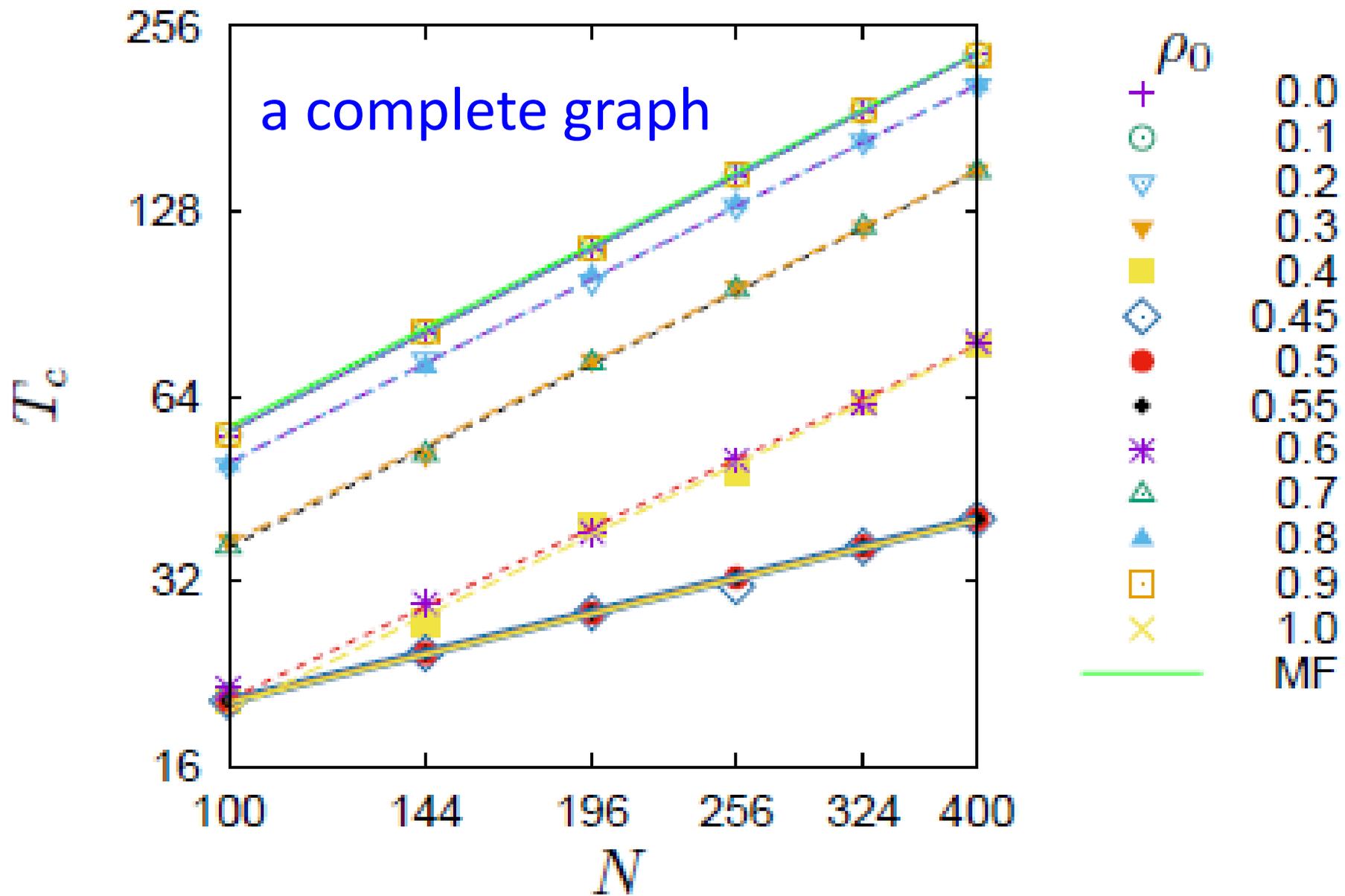


FIG. 3: Time evolution of density $\rho(t)$ of positive links for $N = 256$, (a) $T = 20$, (b) $T = 120$.

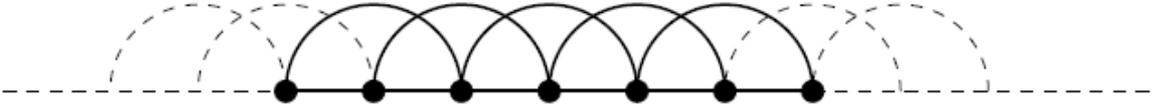
K. Malarz, K. K., Phys. Rev, E 103 (2021) 066301



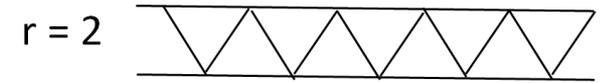
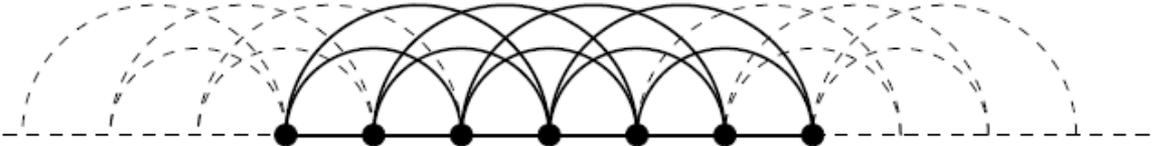
(a) $M = 7, r = 1$



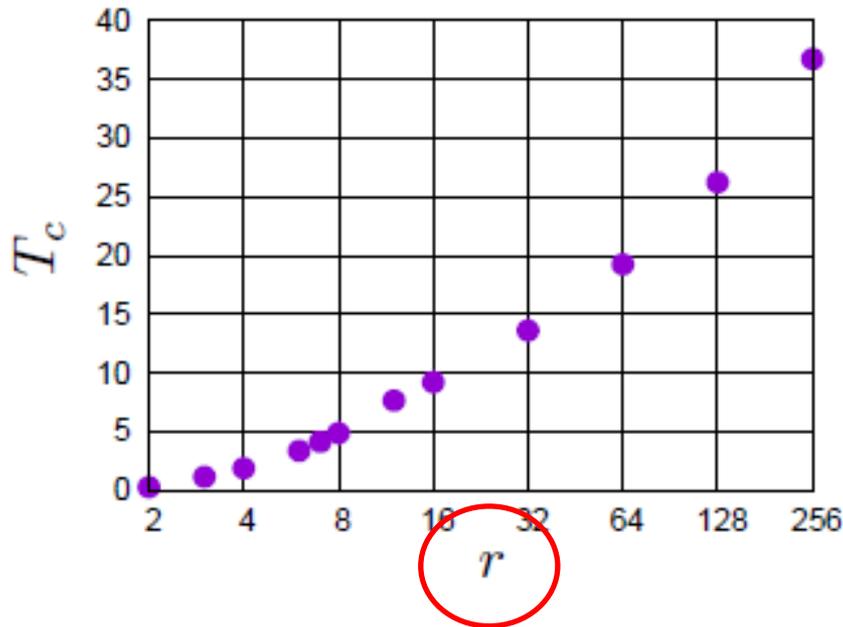
(b) $M = 7, r = 2$



(c) $M = 7, r = 3$



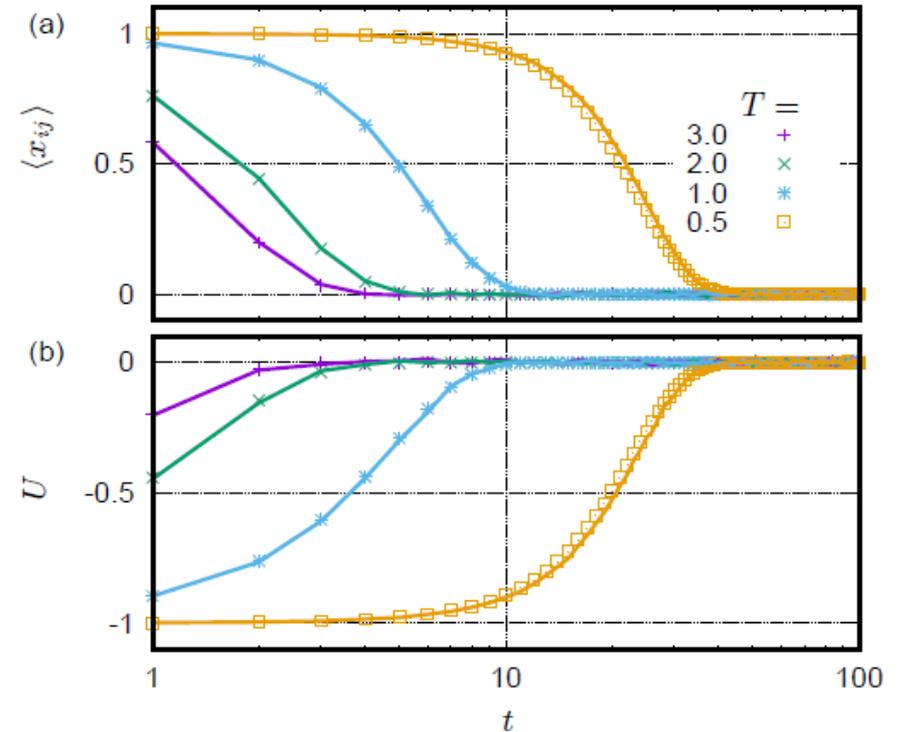
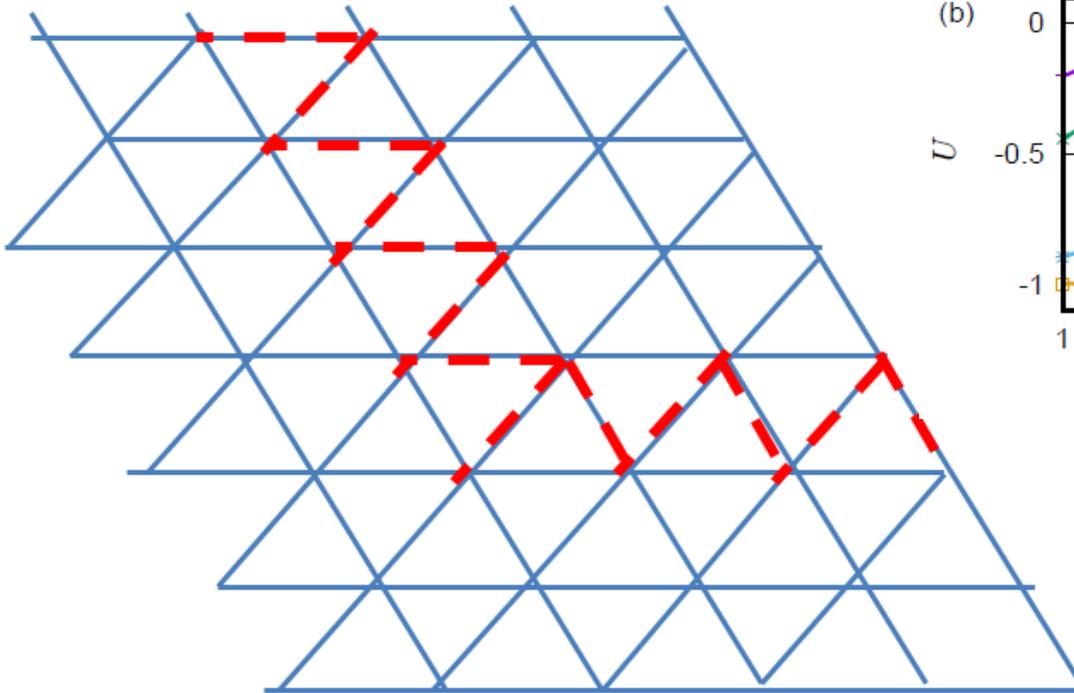
Watts-Strogatz
ordered chain
- no rewiring



The critical temperature
as dependent
on the interaction range r
and a thermal noise

$$T_c \cong 0$$

triangular lattice –
zero-energy domain wall

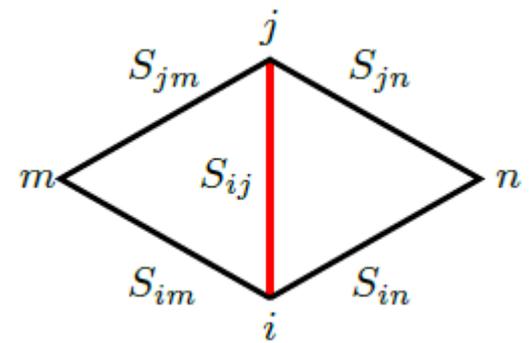


triangular lattice –
U/triad vs time
(lower panel)

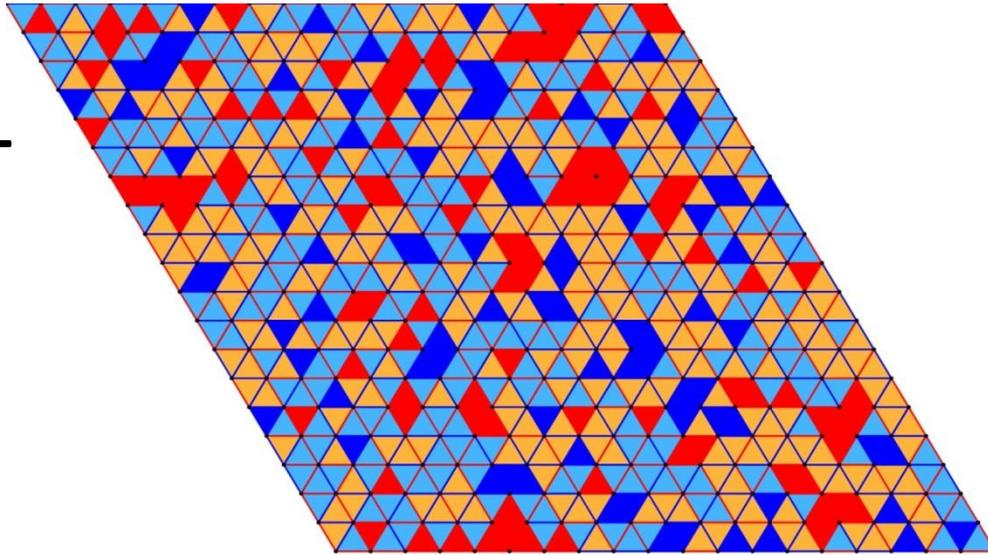
K. Malarz, M. Wołoszyn, Chaos 30, 121103 (2020)

K. Malarz, M. Wołoszyn, K.K., Physica D 411 (2020) 132506

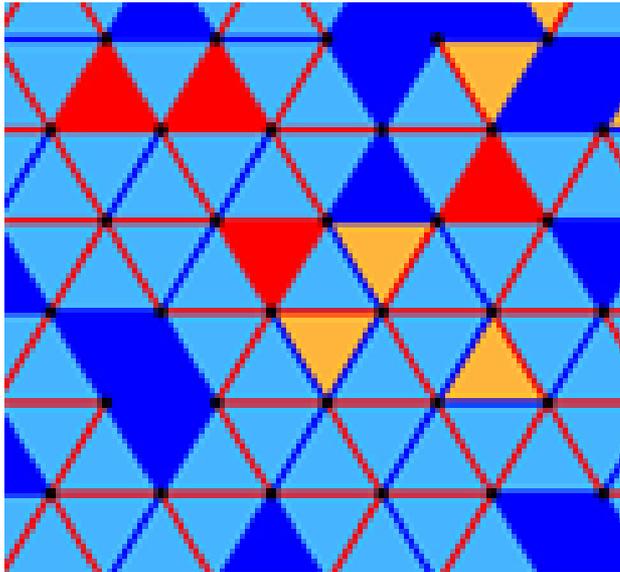
$$s_{ij}(t+1) = \text{sign}[s_{im}(t)s_{mj}(t) + s_{in}(t)s_{nj}(t)]$$



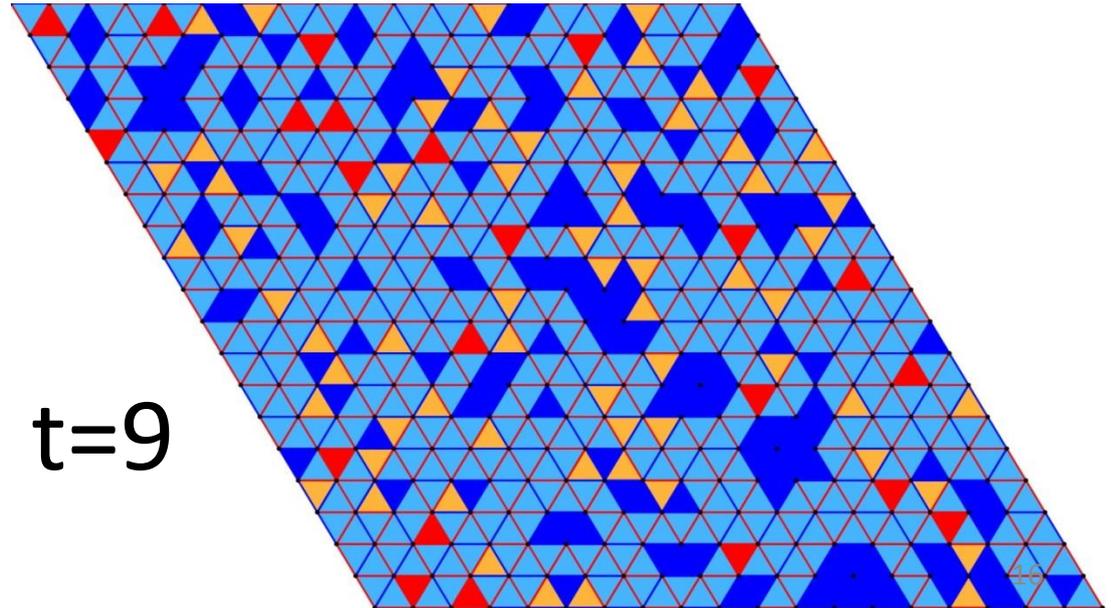
t=1



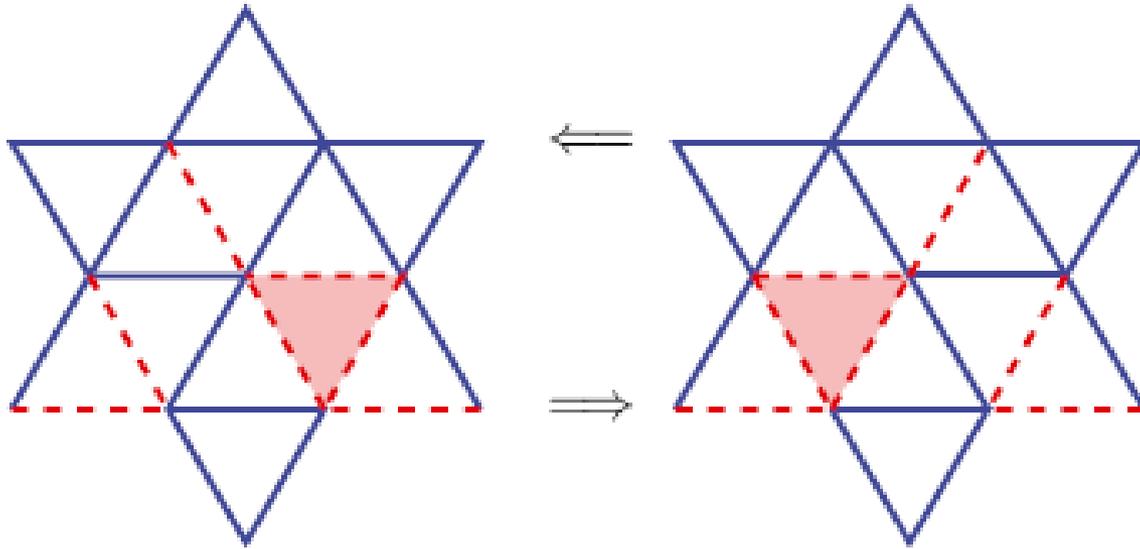
Local neighborhood:
imbalanced steady
configurations



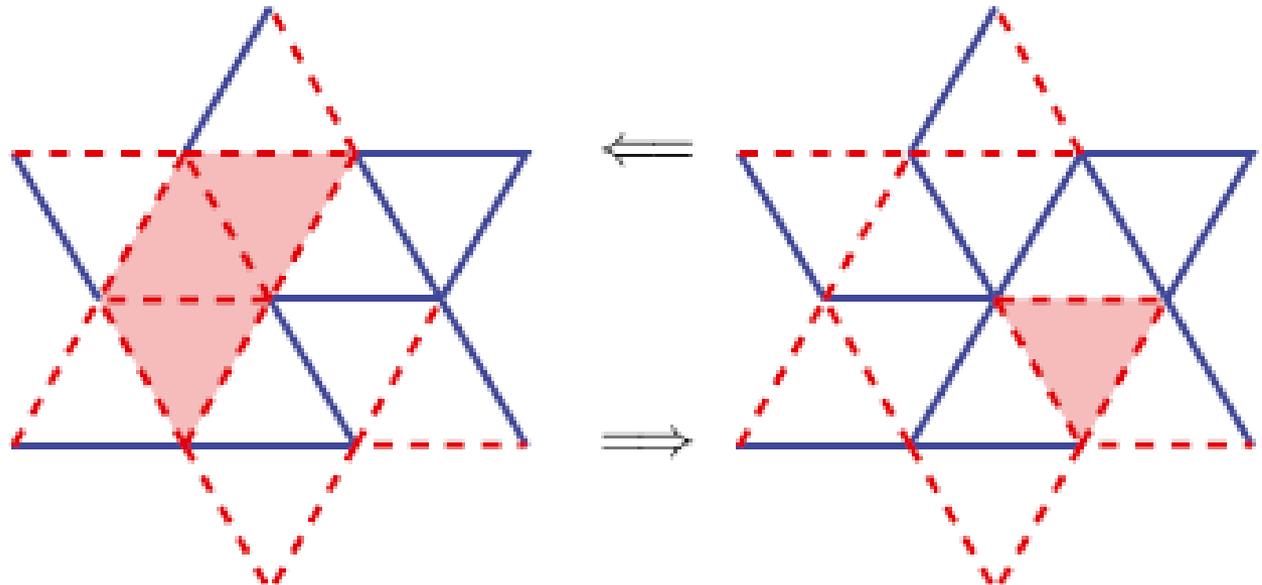
t=9



$$s_{ij}(t+1) = \text{sign}[s_{im}(t)s_{mj}(t) + s_{in}(t)s_{nj}(t)]$$



limit cycles
of length 2

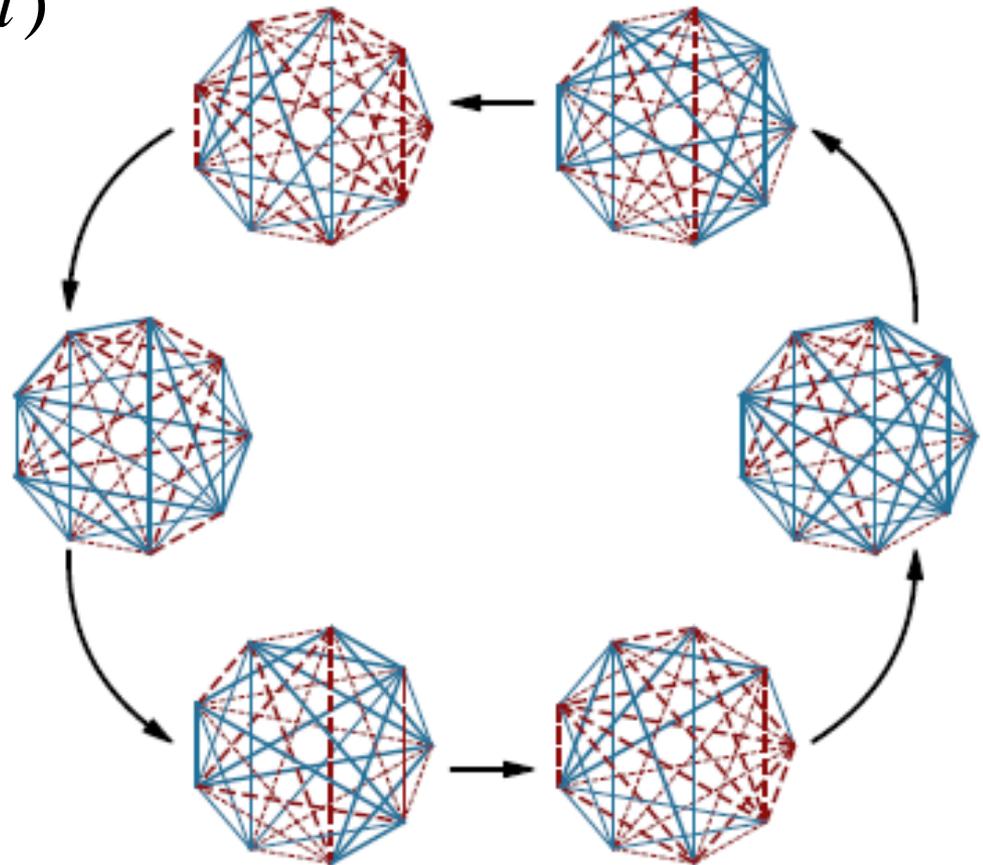


K. Malarz,
M. Wołoszyn, K.K.,
Physica D 411
(2020) 132506

complete graph, synchronous CA,
global neighborhood

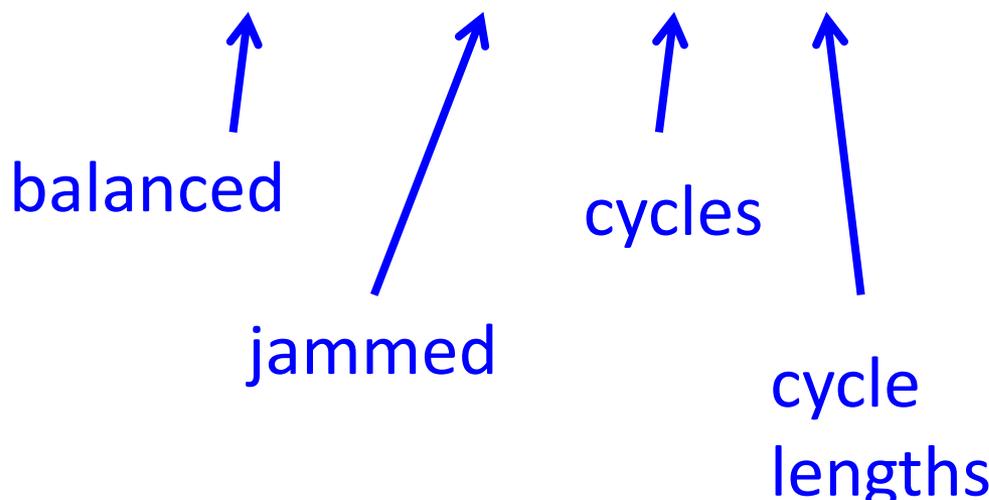
$$s_{ij}(t+1) = \text{sign} \sum_k^{N-2} s_{ik}(t) s_{kj}(t)$$

M. J. Krawczyk, K. K., Z. Burda,
Towards the Heider balance
-- a cellular automaton
with a global neighborhood,
arXiv:2105.04010



N	B	J	C	L_c
5	0.5329	0.0	0.4671	2
6	0.5844	0.2385	0.1771	2
7	0.6302	0.0	0.3698	2,3
8	0.7403	0.0412	0.2185	2,3
9	0.6468	0.0150	0.3382	2,3,4,6,12
10	0.9135	0.0378	0.0487	2,3,4
11	0.8954	0.0030	0.1016	2,3,4,6
12	0.9403	0.0335	0.0262	2
13	0.9329	0.0011	0.0660	2,4
14	0.9494	0.0359	0.0147	2
15	0.9560	0.0008	0.0432	2

N=11	
L_c	fraction
1	0.8987018
2	0.1007685
3	0.0002326
4	0.0001544
6	0.0001416
14	0.0000011



$$\frac{ds_{ij}}{dt} = \sum_k^{N-2} s_{ik} s_{kj}$$

Seth A. Marvel



Jon Kleinberg



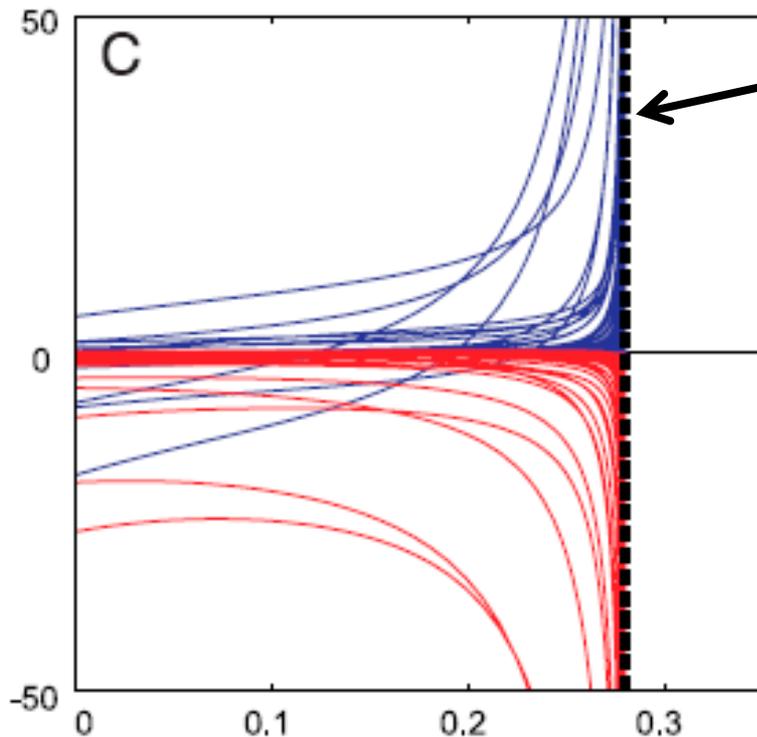
Robert D. Kleinberg



Steven H. Strogatz



17 countries in WWII except Denmark and Portugal

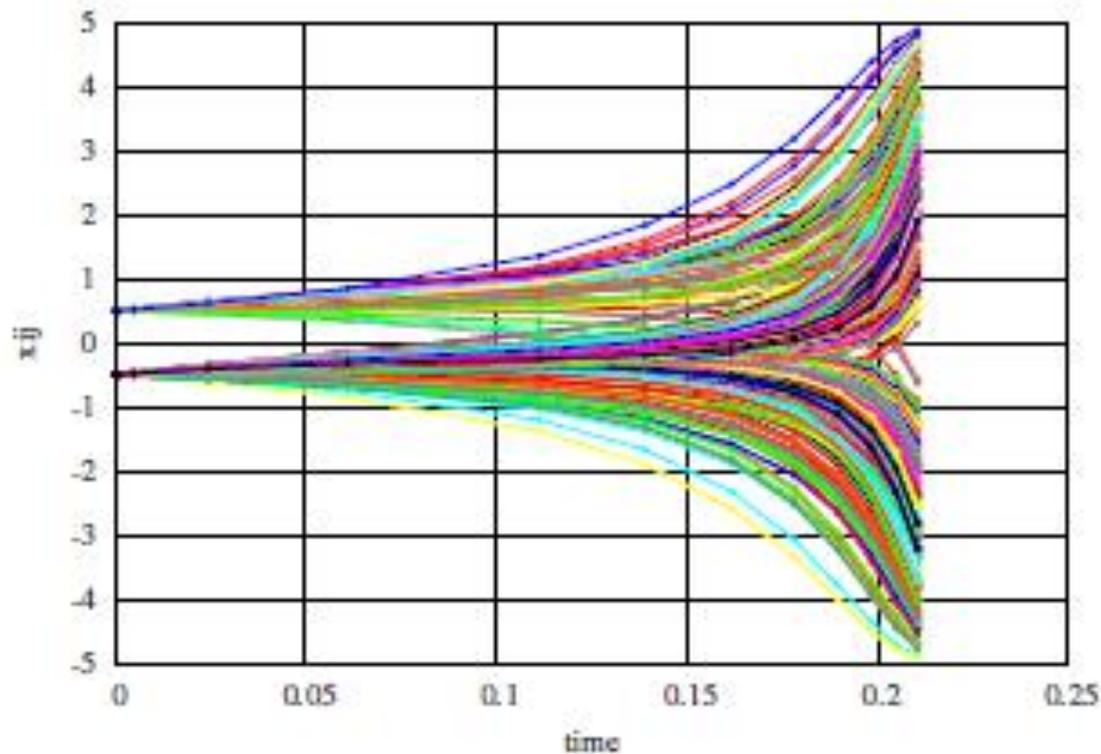


PNAS 108 (2011) 1771

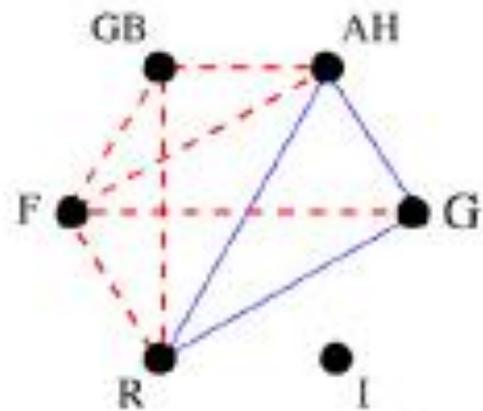
Continuous-time model of structural balance

$$\frac{ds_{ij}}{dt} = \left(1 - \frac{s_{ij}^2}{R^2} \right) \sum_k s_{ik} s_{kj}$$

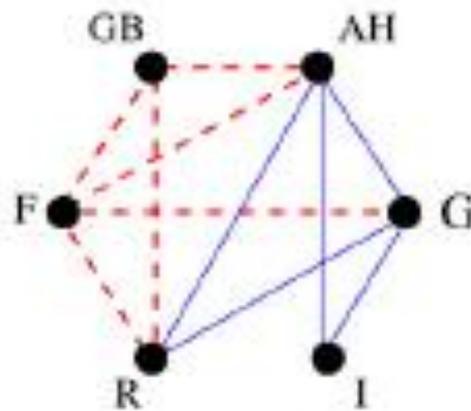
Generic solution
is balanced



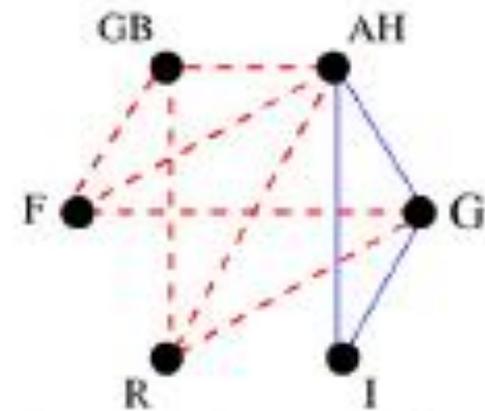
K. K., P. Gawroński, P. Gronek, Int. J. Mod. Phys. C **16** (2005) 707;
P. Gawroński., M. J. Krawczyk, K. K., Acta Phys. Pol. B 46 (2015) 911



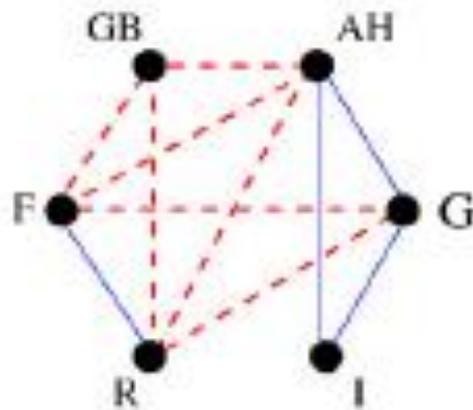
3 Emperor's league 1872–81



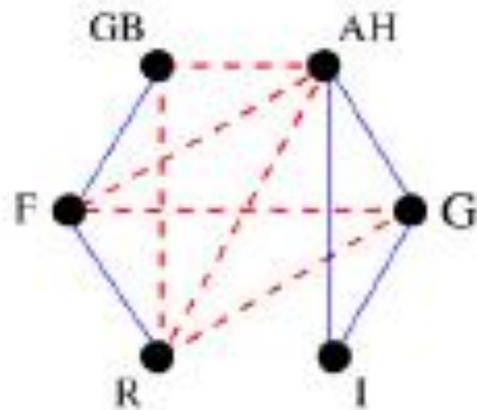
Triple Alliance 1882



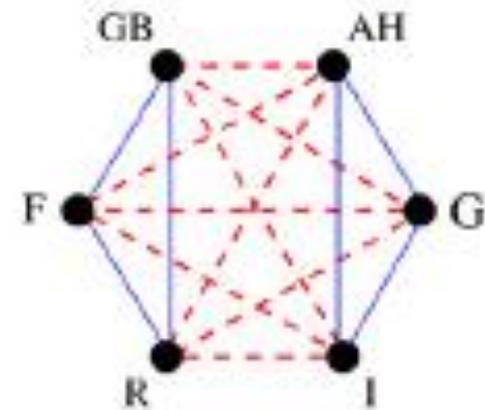
German-Russian Lapse 1890



French-Russian Alliance 1891–94



Entente Cordiale 1904



British-Russian Alliance 1907

Fig. 10. Evolution of the major relationship changes between the protagonists of World War I from 1872–1907.

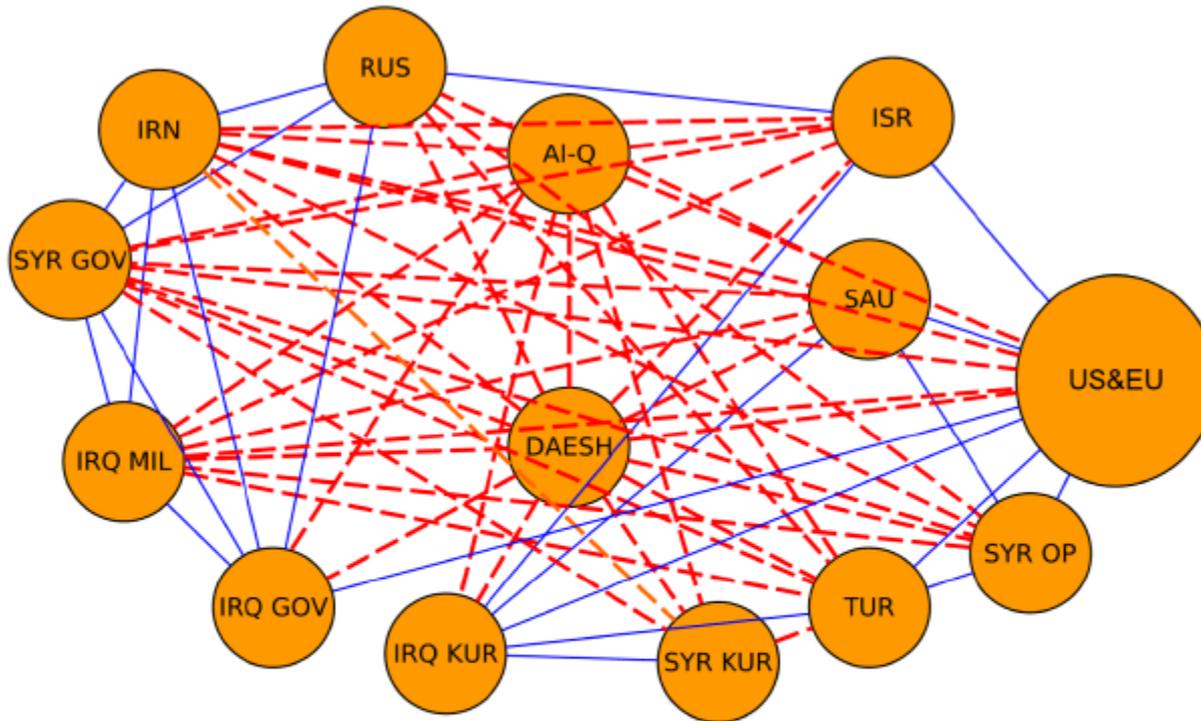


Fig 6. Network of relationships in the Middle East. Graph with the relationships between 21 different “agents” in the Middle East as reported by The Economist in December 2015 [24]. We discriminate between enemies (red), allies (blue) and neutrals (no link). We use the following abbreviations to specify the nodes: Moderate Sunni Arab opposition (SYR OP); Syrian government (SYR GOV); Syrian Kurds (SYR KUR); Iraqi Kurds (IRQ KUR); Iraqi Shia militias (IRQ MIL); Iraqi government (IRQ GOV); Turkey (TUR); Israel (ISR); Russia (RUS); USA and E.U (US&EU); Saudi Arabia & Arab League (SAU); Iran (IRN); Al-Qaeda/Jabhat al-Nusra (AL-Q); DAESH or ISIS (DAESH). The node “US&EU” represents 8 agents (US, Australia, Belgium, Canada, France, Germany, Netherlands and United Kingdom).

How far is the final balanced state from the balanced state most close to the initial one?

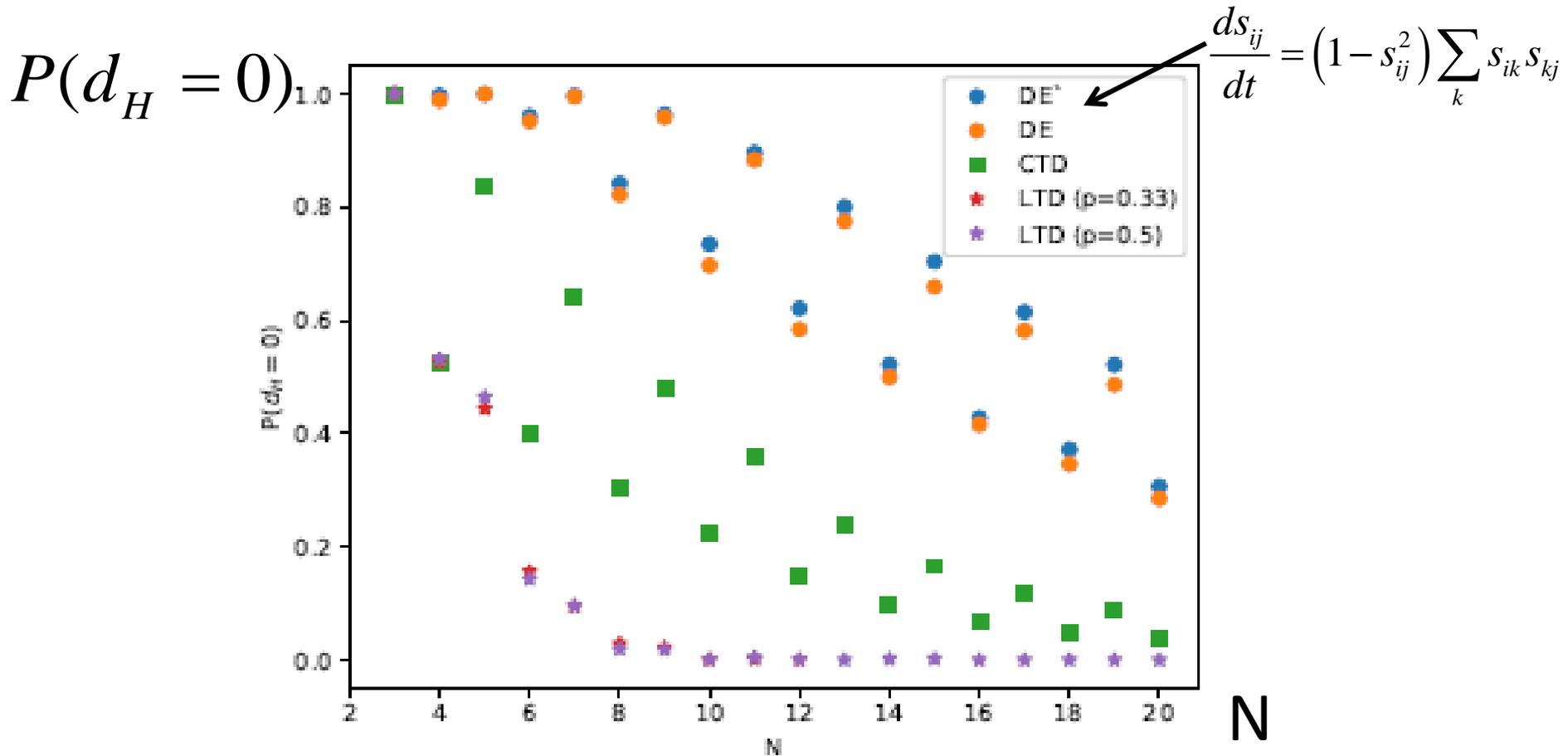


Figure 3. The probability $P(d_H = 0)$ that the final (balanced) state obtained by particular algorithms is just the same as the balanced state most close to the initial state.

M. J. Krawczyk, K. K., Structural balance of opinions, *in preparation*

Data collected by the National Opinion Research Center in 44 American states in 1974-1988 [P. Brace et al., Amer. J. Political Science, 46 (2002) 173]

1. Tolerance (more tolerance)
2. Race (less racist)
3. Abortion (more pro-choice)
4. Religiosity (more religious)
5. Homosexuality (more acceptant of homosexuality)
6. Public feminism (more acceptant of women's rights)
7. Environment spending (support higher government spending on the environment)
8. Welfare spending (support higher government spending on welfare)
9. Death penalty (support for death penalty)



support(US state)
->Pearson $r(i,j)$

1	0.907	0.686	-0.551	0.828	0.832	0.337	-0.151	0.163
	1	0.565	-0.443	0.828	0.887	0.414	-0.055	0.296
		1	-0.725	0.776	0.600	0.423	-0.219	-0.052
			1	-0.694	-0.571	-0.201	0.277	0.111
				1	0.760	0.354	-0.174	-0.155
					1	0.296	-0.277	0.266
						1	0.080	-0.140
							1	-0.042
								1

The symmetric matrix $r(i,j)$ of Pearson correlations of opinions on nine issues, collected in Ref. ²⁶.

The data handling:

$$s_{ij}(t=0)=r(i,j) + \frac{ds_{ij}}{dt} = \left(1 - s_{ij}^2\right) \sum_k s_{ik} s_{kj} \Rightarrow \text{a balanced state}$$

Result - **portraits of voters:**

4,8 vs 1,2,3,5,6,7,9

=religiosity + welfare spending vs other issues

is interpretable within

the Comfort Hypothesis* by Earl Babbie:

„Parishioners whose life situations most deprive them of satisfaction and fulfillment in the secular society turn to the church for comfort and substitute rewards.”



* E. Babbie, *The Practice of Social Research*, 2007

Concluding, the following results have been highlighted:

- role of topology and initial states,
- long limit cycles,
- a new application to social opinions

and the related problems are:

- why $T_c=0$ for triangular lattices?
- what the long cycles mean?
- what other applications?

THANK YOU

TABLE 1 State Sample Sizes and State-Level Opinions from the GSS, 1974–1998.*

	Sample size	GSS PSUs	Tolerance (rank)	Race (rank)	Abortion (rank)	Religiosity (rank)	Homosexuality (rank)	Public Feminism (rank)	Environment Spending (rank)	Welfare Spending (rank)	Death Penalty (rank)
Alabama	872	9	.42 (44)	.50 (44)	3.06 (41)	.68 (2)	.04 (44)	1.01 (43)	.68 (40)	.43 (4)	.67 (40)
Alaska	57	1	.80 (1)	.87 (2)	4.17 (9)	.45 (43)	.37 (2)	1.63 (5)	.73 (22)	.30 (38)	.66 (42)
Arizona	515	5	.65 (11)	.80 (14)	3.99 (15)	.48 (40)	.23 (16)	1.51 (13)	.71 (35)	.31 (33)	.80 (6)
Arkansas	288	4	.44 (41)	.52 (41)	3.42 (37)	.62 (6)	.10 (36)	1.01 (44)	.68 (43)	.40 (9)	.68 (37)
California	3471	24	.67 (9)	.61 (7)	4.50 (1)	.45 (41)	.29 (5)	1.52 (12)	.75 (15)	.35 (20)	.76 (12)
Colorado	911	8	.67 (10)	.80 (10)	4.03 (12)	.49 (37)	.27 (8)	1.57 (8)	.71 (34)	.34 (25)	.74 (19)
Connecticut	574	7	.69 (4)	.80 (13)	4.35 (5)	.54 (24)	.27 (7)	1.60 (7)	.79 (5)	.30 (40)	.69 (29)
Delaware	74	2	.56 (32)	.79 (15)	3.93 (19)	.54 (23)	.23 (14)	1.65 (3)	.80 (3)	.26 (44)	.68 (39)
Florida	1549	14	.56 (33)	.69 (34)	3.97 (16)	.54 (22)	.18 (23)	1.36 (34)	.73 (20)	.33 (29)	.77 (8)
Georgia	883	9	.53 (38)	.65 (37)	3.54 (35)	.63 (4)	.10 (38)	1.28 (37)	.70 (38)	.36 (15)	.74 (17)
Illinois	1389	15	.62 (19)	.74 (23)	4.01 (14)	.56 (18)	.22 (17)	1.47 (21)	.78 (8)	.44 (2)	.75 (14)
Indiana	818	10	.59 (26)	.69 (35)	3.56 (34)	.57 (16)	.13 (34)	1.43 (26)	.76 (12)	.34 (27)	.74 (16)
Iowa	267	2	.61 (20)	.77 (20)	3.93 (20)	.55 (21)	.18 (24)	1.47 (20)	.75 (14)	.37 (14)	.67 (41)
Kansas	439	6	.63 (15)	.72 (27)	3.91 (21)	.56 (17)	.19 (20)	1.46 (23)	.72 (27)	.33 (28)	.73 (20)
Kentucky	269	3	.43 (42)	.52 (42)	2.91 (42)	.51 (32)	.05 (43)	1.12 (41)	.71 (31)	.32 (32)	.64 (44)
Louisiana	448	5	.53 (36)	.72 (28)	3.25 (40)	.58 (12)	.17 (26)	1.33 (36)	.78 (9)	.43 (5)	.70 (26)
Maryland	530	7	.63 (14)	.80 (9)	4.36 (4)	.53 (29)	.25 (12)	1.49 (15)	.77 (10)	.42 (6)	.70 (25)
Massachusetts	706	6	.68 (6)	.80 (12)	4.27 (7)	.51 (35)	.32 (3)	1.57 (11)	.79 (4)	.34 (26)	.69 (30)
Michigan	1787	16	.61 (22)	.75 (22)	3.69 (28)	.56 (20)	.19 (21)	1.46 (22)	.73 (24)	.31 (35)	.68 (36)
Minnesota	723	6	.62 (18)	.75 (21)	3.57 (33)	.60 (10)	.21 (19)	1.49 (17)	.73 (23)	.29 (41)	.65 (43)
Mississippi	251	2	.53 (37)	.73 (26)	2.88 (43)	.60 (9)	.10 (37)	1.45 (25)	.69 (39)	.46 (1)	.70 (27)
Missouri	947	10	.58 (27)	.71 (31)	3.65 (29)	.57 (14)	.13 (32)	1.41 (28)	.74 (17)	.39 (10)	.75 (13)
Montana	176	2	.58 (29)	.70 (32)	4.06 (11)	.53 (27)	.17 (25)	1.45 (24)	.68 (41)	.30 (39)	.82 (3)
New Hampshire	172	1	.68 (7)	.81 (8)	4.12 (10)	.51 (34)	.24 (13)	1.57 (10)	.85 (1)	.36 (16)	.76 (10)
New Jersey	1145	12	.62 (17)	.77 (18)	4.24 (8)	.52 (31)	.25 (11)	1.49 (16)	.78 (7)	.30 (37)	.71 (22)
New York	2432	22	.63 (13)	.80 (11)	4.35 (6)	.51 (33)	.27 (9)	1.48 (18)	.78 (6)	.37 (13)	.68 (34)
North Carolina	1161	12	.45 (40)	.55 (40)	3.50 (36)	.63 (3)	.09 (39)	1.17 (40)	.72 (25)	.32 (31)	.70 (28)
North Dakota	271	2	.57 (30)	.78 (17)	3.41 (38)	.61 (8)	.16 (27)	1.49 (14)	.72 (29)	.34 (24)	.75 (15)
Ohio	1730	17	.58 (28)	.68 (36)	3.74 (26)	.52 (30)	.14 (30)	1.41 (30)	.72 (26)	.34 (21)	.76 (9)
Oklahoma	411	4	.57 (31)	.74 (24)	3.70 (27)	.53 (26)	.15 (29)	1.42 (27)	.73 (21)	.34 (23)	.81 (5)
Oregon	481	5	.69 (3)	.84 (4)	4.40 (3)	.45 (42)	.23 (15)	1.57 (9)	.75 (16)	.41 (7)	.71 (24)
Pennsylvania	1744	16	.60 (25)	.70 (33)	3.87 (24)	.57 (15)	.16 (22)	1.38 (33)	.76 (11)	.31 (36)	.78 (7)
Rhode Island	146	1	.67 (8)	.88 (1)	4.02 (13)	.54 (25)	.27 (6)	1.47 (19)	.84 (2)	.44 (3)	.71 (23)
South Carolina	504	6	.61 (21)	.77 (19)	3.93 (19)	.49 (38)	.26 (10)	1.79 (1)	.71 (32)	.29 (42)	.82 (4)
South Dakota	68	1	.54 (35)	.63 (38)	3.88 (23)	.62 (5)	.13 (31)	1.25 (38)	.74 (18)	.35 (19)	.73 (21)
Tennessee	1040	11	.49 (39)	.60 (39)	3.58 (32)	.62 (7)	.08 (40)	1.21 (39)	.70 (36)	.38 (11)	.69 (31)
Texas	1912	19	.56 (34)	.71 (29)	3.63 (30)	.59 (11)	.13 (33)	1.40 (32)	.73 (19)	.34 (22)	.76 (11)
Utah	192	1	.65 (12)	.85 (3)	2.78 (44)	.75 (1)	.08 (41)	1.41 (29)	.70 (37)	.40 (8)	.96 (1)
Vermont	110	2	.68 (5)	.78 (16)	3.96 (17)	.48 (39)	.42 (1)	1.67 (2)	.68 (44)	.35 (17)	.68 (33)
Virginia	1285	14	.61 (23)	.71 (30)	3.88 (22)	.56 (19)	.21 (18)	1.34 (35)	.75 (13)	.37 (12)	.69 (32)
Washington	527	6	.71 (2)	.82 (6)	4.41 (2)	.43 (44)	.29 (4)	1.60 (6)	.68 (42)	.35 (18)	.74 (18)
West Virginia	479	5	.43 (43)	.50 (43)	3.37 (39)	.53 (28)	.07 (42)	1.02 (42)	.72 (30)	.33 (30)	.68 (36)
Wisconsin	946	8	.63 (16)	.74 (25)	3.62 (31)	.58 (13)	.16 (28)	1.40 (31)	.71 (33)	.26 (43)	.68 (38)
Wyoming	146	1	.61 (24)	.82 (5)	3.82 (25)	.51 (36)	.12 (35)	1.64 (4)	.72 (28)	.31 (34)	.93 (2)
Mean			.60	.73	3.80	.55	.19	1.43	.74	.35	.73
Standard Dev.			.08	.10	.42	.06	.08	.17	.04	.05	.07

*Hawaii, Idaho, Maine, Nebraska, Nevada, and New Mexico are not included in this analysis.

Note: The method for computing these variables is described in Appendix 1a.

$a(s, i)$ - support for issue i in state s , averaged over years

$\bar{a}(i)$ - support for issue i , averaged over states

$$r(i, j) = \frac{\sum_s (a(s, i) - \bar{a}(i))(a(s, j) - \bar{a}(j))}{\sqrt{\sum_s (a(s, i) - \bar{a}(i))^2} \sqrt{\sum_s (a(s, j) - \bar{a}(j))^2}}$$

Example:

	God	Money	Sex
Ohio	7	9	3
Idaho	3	1	7
Iowa	8	7	2
Utah	2	3	8
mean	5	5	5
st. dev.	2,5	3	2.5

$$r(\text{God}, \text{Money}) = \frac{1}{3 \times 2,5} [2 \times 4 + (-2) \times (-4) + 3 \times 2 + (-3) \times (-2)] \approx 3,7$$

$$r(\text{God}, \text{Sex}) = \frac{1}{2,5 \times 2,5} [2 \times (-2) + (-2) \times 2 + 3 \times (-3) + (-3) \times 3] \approx -4,2$$

